Proportion in Bach's Air from the French Suite in C minor

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Background in music theory/analysis/composition. The concept of proportion is inherently related to the fundamental musical materials and to the aesthetic ideals of beauty and perfection. It is evident that music can only result from movements and sounds measured by numbers. On the contrary, that which is devoid of number lacks any sort of reason; it is without order, without beauty, without grace and ultimately deprived of all perfections (Plato, *Epinomis*, 977d.). These ideas had a strong influence on the aesthetic view of the baroque era; particularly in music, they constituted the ground for the connection of proportions with formal unity, symmetry, and balance. In the compositional process proportions function as a mean of achieving these qualities in the organization of the time space of the musical structure; then, a series of questions rises: why they are there, how they are related to the musical form, how they are manifested in the written music, how they are perceived during listening, if the composer used them intentionally or not, etc. Concerning the use of proportions in Bach's works, various interpretive perspectives have emerged. Among them, a well-evidenced one is the theory of proportional parallelism which points out that proportions and numbers in Bach's works demonstrate the qualities of unity and strong coherence of his music, regardless of whether they were designed by Bach or not. The examination of the proportional relations refers to a large amount of music, thus, it provides information for the organization of a large-scale level of musical time (Tatlow, 2007). Starting from this perspective, this study attempts a step to the opposite direction. It focuses on a single piece of sixteen measures and follows the compositional structure step by step. This analysis attempts to enter the heart of the composition in order to provide a unified interpretation of its individual features.

Background in mathematics. Music and mathematics are literary connected to each other through the use of the concepts of ratios, special functions, statistics, and geometry (Hodges, 2006). Especially, ratios and functions are associated to proportions and processes that may appear in a musical composition. Thus, they constitute powerful devices for musical analysis, in the sense that they might demonstrate aspects of the relationships of the musical elements which otherwise would not be approachable. These structural relationships may be represented geometrically by a graph, which is conceptually a spatial configuration with a finite set of points, called vertices, and a finite set of lines, called edges, joining one point to another. The geometric drawing concerns include the exact coordinates in a two-dimensional plane of the images of the vertices and the edges which have a special meaning. An orthogonal drawing is a drawing in which each edge is represented by a chain of horizontal and vertical line segments (CRC Handbook of discrete and combinatorial mathematics, 2000). Particularly, assuming that the x axis is time and the y axis comprises some values of the function $f(n)=1:2^n$, where n is some integer, each vertex represents the change of some musical event and the horizontal part of each edge is assigned a number, called the weight, that has the meaning of the duration of that event. In this paper, the purpose of such drawings is the graphical representation of the structural relationships between the microscopic and the macroscopic musical events so as to depict the formal process of the work and to contribute to its comprehension.

Aims. The paper explores how proportions are involved in the formal process of the work. This is accomplished through the examination of the following components: the harmonic rhythm, the structural rhythm, and the overall form.

Main contribution. The preceding analysis demonstrates how proportional relations influence the generation of the musical structure. They may be grouped in two general types: 1) proportions that are manifested in the relation of measure numbers and they are explicit on the score; 2) proportions that are manifested in processes underlying the musical surface and come to light only by deep and detailed observation. Proportions of the first type exist in many structural levels of the work: the individual parts, the formal sections, and the overall form. Proportions of the second type exist in two constructional axis of the formal process which are represented by the function $f(n)=1:2^n$: the alteration of the harmonic rhythm, as an expression of how the harmonic relations are carried out through time, and the alteration of the structural rhythm, as an expression of how the overall form is being constructed by the displacement of structural units. These relations are not isolated events; they are supported by reflections of harmonic

and melodic elements. The proportions of the work function in both time directions, forward and backward, as a result of the retrograde process that originates the formal sections of the piece. The two types of proportional relations mentioned above are in complete interaction: if proportions of the second type exist in the formal process then proportions of the first type will be a natural consequence; on the other hand if proportions of the first type exist in the written music, this indicates that proportions of the second type contribute to the formal process. In respect to the piece discussed here, proportional relations are a principle of reason for the explanation of the form of the piece. They provide a reasonable cause so that we may approach the work in order to achieve a better understanding of it. So they represent an image of the ultimate expression of unity, symmetry and balance between the whole and its individual parts. This interpretation indicates a high formal design, a well designed compositional plan, and in turn, a scientific compositional view.

Implications. The main point of the present analysis is that points of microscopic changes which take place in the piece are treated as individual worlds that, regardless of how small they are, may have a strong impact to the macrocosm of the piece. As shown in this study, this feature was of great significance because it enabled the analysis to reach a unified result. Consequently, proportions are involved in progressive and retrogressive processes of the organization of the musical time which result in the generation of the overall form out of a basic initial material. Under this perspective, the proportions that are explicit in the numbers of measures of the formal sections are the natural outcome of the existence of proportions operating as motivating factors of those processes. This study demonstrates an aspect of constructional analysis that may function as interpretive principle for other pieces as well.

Additionally, the analysis touches some of the most significant issues in Bach's studies: Bach's compositional process and the compositional planning, the conception of musical space and time, the unity of content and form, and the scientific nature of his mind. It is mostly related to the theory of proportional parallelism mentioned above, especially to the proportion level 1 that refers to proportions between sections of a movement (Tatlow, 2007). Targeting to the microcosmic events of the musical structure, this study might add lower levels of proportions within a movement.

Provided that proportion is the relation of two terms to each other[i] and that all forms of music have organized time as a fundamental property[ii], the following analysis focuses on how the musical time of the piece is organized by demonstrating the relations between events of changes and time spaces. The analysis is based on simple successive reductions that include the removal of non-chord tones, of repeated tones, and of chord inversions. In this way, the contrapuntal texture of the piece is simplified to a kind of an abstract sketch: the harmonic background, represented by a fundamental base that carries the tonal functions.[iii] Then, the analysis examines the harmonic and structural relationships demonstrated in this fundamental level of the composition. The consideration of these relationships is based on the definition of the following concepts.

The concept of *harmonic rhythm* (HR) defines the duration of harmony in time as follows: HR=harmony/time, where harmony=a scale degree expressed by its tonal function and time=the number of the occupied time units; time unit=a quarter-value note. In this sense, each scale degree, along with its duration, represents a harmonic unit.

Similarly, the concept of *structural rhythm* (SR) expresses the duration of the structural units in time; considering that a structural unit is a repeated structure-pattern that combines both harmonic units and melodic elements, SR=structural unit/time, where time= the number of the occupied time units; time unit=a quarter-value note.

According to these definitions both HR and SR might be conceived as processes that work in the musical space of the piece. This study examines how they are related and how they contribute to the creation of the overall form. This is achieved through the observation of the points where these processes change[iv] which are represented by series of fractions that define CHR and CSR correspondingly.[v]

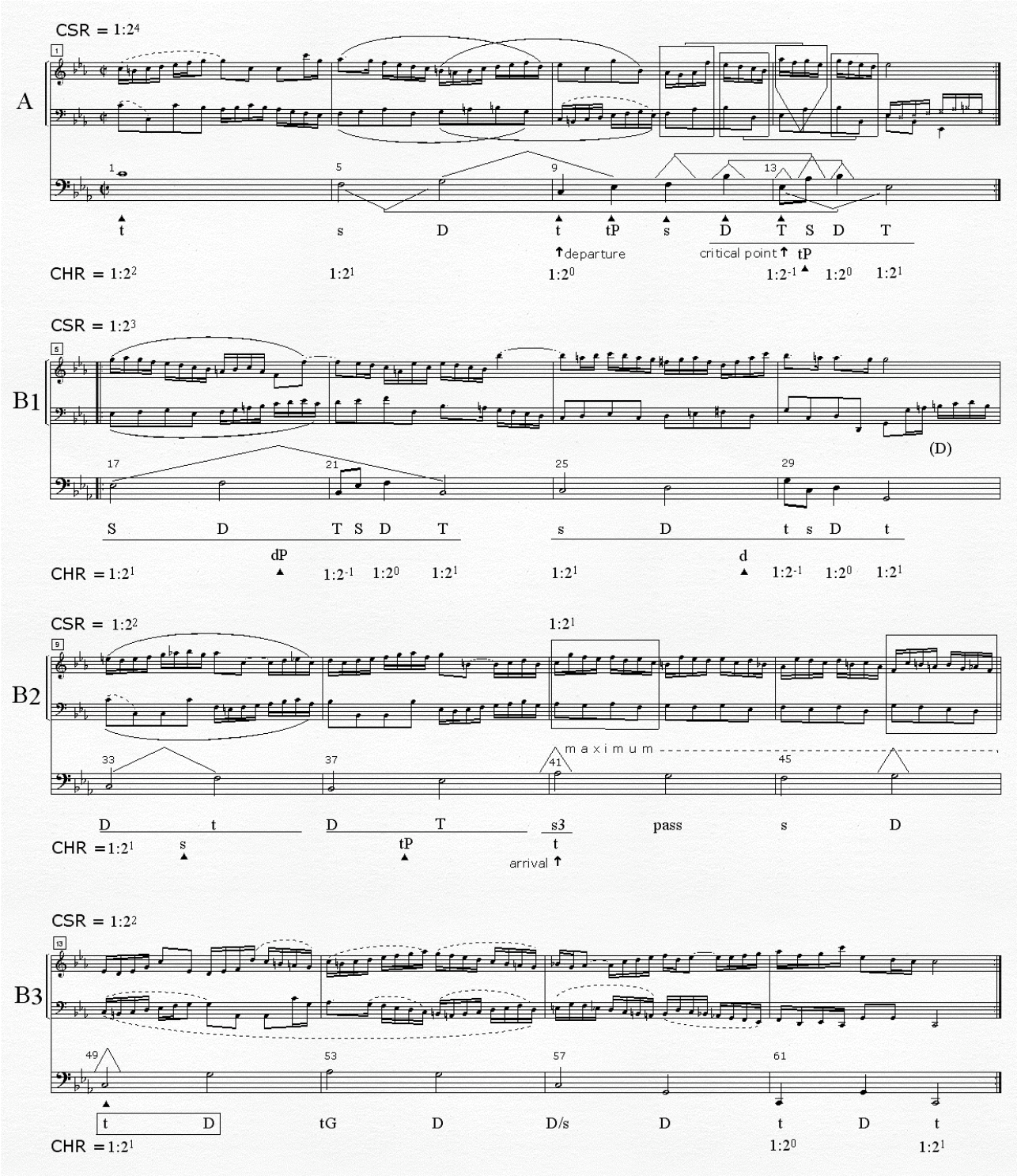


Figure 1. Bach's Air from the French Suite in C minor: Staves 1-2 are the actual music of the piece, measured in measure numbers. Staff 3 is the fundamental base with the tonal functions, measured in time units, time unit=a quarter-value note; the numbers below the tonal functions represent the points of the changes of the harmonic rhythm, CHR. The numbers above staff 1 represent the point of the changes of the structural rhythm, CSR. Lines and shapes refer to the relations of the structural units; dashed slurs indicate some of the recurrences of the prime motive and its altered forms; black triangles and up-arrow markings demonstrate the harmonic reflections.

Form and harmony

A first glance at the piece reveals a binary form, where part A consists of four measures and part B of twelve[vi]. Moreover, the internal structure of part B is divided into three four-measure subparts, presented respectively as B_1, B_2, B_3, so that the overall form is represented as shown: Part A: mm. l-4, Part B: B_1: mm. 5-8, B_2: mm. 9-12, B_3: mm. l3-16. Figure 1 provides an annotated copy of the score[vii] that demonstrates the fundamental base with the tonal functions in the bottom staff, the terms of CHR at the bottom staff and of CSR at the top staff of each system, as well as special markings on the relation of the structural units and of the harmonic reflections.[viii]

The opening part as an exposition: A (mm. 1-4)

The way in which the harmonic progressions appear indicates a gradual alteration of the harmonic rhythm (Figure 1). So, the changes of HR (CHR) are as shown in Table 1.

Part	Measure /time unit	Harmony /tonal function	Duration	CHR
A	1/1	1/t	4	1:4 or $1:2^2$
	2/5	1/s-D	2	1:2 or $1:2^1$
	3/9	1/t-tP-s-D	1	1:1 or $1:2^0$
	4/13	1/T-S	½	1:1/2 or $1:2^{-1}$
	4/14	1/D	1	1:1 or $1:2^0$
	4/15	1/T	2	1:2 or $1:2^1$

Table 1. The changes of the harmonic rhythm, CHR, in part A.

Hence, CHR is represented by the following terms $1:2^2$, $1:2^1$, $1:2^0$, $1:2^{-1}$, $1:2^0$, $1:2^1$ of the discrete exponential function $f(n)=1:2^n$, where n is some integer. This interpretation leads to interesting observations.

First, the set of the above terms is divided into two subsets, $CHR_1 = \{1:2^2, 1:2^1, 1:2^0, 1:2^{-1}\}$ and $CHR_2 = \{1:2^{-1}, 1:2^0, 1:2^1\}$, that have $1:2^{-1}$ in common (Figure 2a). In CHR_1 each term leads to the following one by multiplication by 2; thus, each term is related to the following one by the proportion 1:2. This proportional increase results to a proportional acceleration of the harmonic rhythm. In CHR_2 each term leads to the following one by division by 2 and, consequently, each term is related to the following one by the proportion 2:1. This proportional decrease results to a proportional retardation of the harmonic rhythm. Additionally, when part A is repeated, retardation continues to the term $1:2^2$, thus, the first term in CHR_1. Consequently, the retardation of CHR_2 leads to the acceleration of CHR_1, as if the two processes define a complete circle. This means that CHR in part A is an underlying cyclic process that comprises the proportions 1:2 and 2:1 as a result of the motion of the harmonic rhythm. The *critical point* of CHR, the common term $1:2^{-1}$, on the first beat of m. 4, divides the four-measure part A into two sections, mm. 1-3 and m.4, related by the proportion 3:1. This is the point where the process of augmentation is reversed and becomes a process of diminution. Additionally, the pitch-material is transferred from the area of the tonic (t) to the area of the relative key (tP).

Second, the duration of the terms of CHR_1, $1:2^2$, $1:2^1$, $1:2^0$, and the duration of the terms of CHR_2, $1:2^{-1}$, $1:2^0$, $1:2^1$, are proportionally related by the ratio 2:1 under retrogression (Figure 2b). Particularly, the first term of CHR_1, $1:2^2$, is related to the third term of CHR_2, $1:2^1$, the second term of CHR_1, $1:2^1$, is related to the second term of CHR_2, $1:2^0$, and the third term of CHR_1, $1:2^0$ is related to the first term of CHR_2, $1:2^{-1}$. The relation 2:1 under retrogression is associated to two things. First, it demonstrates that retardation takes place two times faster than acceleration. Second, it refers to the establishment of the two characteristic tonal areas in part A (Figure 1): the area of the tonic, described by the progression t-(s-D)-t which

corresponds to the three above terms of CHR_1, and the area of the parallel tonic, described by the progression (T-S)-D-T which corresponds to the three above terms of CHR_2. Thus, retardation establishes the parallel tonic area in the half of the time that acceleration establishes the tonic area. This is actually manifested on the score as the relation 2:1 between mm. 1-2 (actually, first beat of m. 3) where the tonic is established, and m. 4, where the parallel tonic is established. Then m. 3 that corresponds to the intermediate space between the term $1:2^0$ of CHR_1 and the term $1:2^{-1}$ of CHR_2 is the space of the transition; the harmonic progression appearing in it, tP-s-D/tP, shows the path of moving away from the tonic.[ix]

Furthermore, one more proportional relation exists in part A, the relation 1:1 between mm. 1-2 and mm. 3-4 (Figure 1). The prime motive in m. 1 appears in the upper voice and a slightly altered form of it appears again in the lower voice in m. 3. These appearances of the prime motive divide A into two smaller sections related by 1:1.

In conclusion, the internal structure of part A demonstrates simultaneously three kinds of proportional relations manifested in the relation of measure numbers: 3:1, 2:1, and 1:1. Each relation is marked by a distinguished event; 3:1 is associated to the reversal of the process of acceleration of HR to retardation; 2:1 is associated to the establishment of the tonic (t) and the parallel tonic (tP); 1:1 is associated to the occurrences of the prime motive. Moreover, the process HR comprises the relations 1:2 and 2:1 as motivating factors within the individual elements of acceleration and retardation correspondingly, and the relation 2:1 as the relation of the interaction between the acceleration and retardation under retrogression.

The contrasting part as development and recapitulation: B (mm. 5-16)

The development consists of two sections, B_1 in mm. 5-8 and B_2 in mm. 9-12.[x]

B_1 (mm. 5-8) demonstrates an internal structure that combines familiar elements (Figure 1). Specifically, a two-measure structural unit is defined in mm. 5-6. The structural unit is primarily exposed in the area of B flat major (dP) in mm. 5-6 and is then repeated in the area of G minor (d) in mm. 7-8. The tonal areas are associated by transposition of a minor third down and the overall tonal space is that of the minor dominant (d). The internal structure points out the relation 1:1.

It is also noticeable that the cadence in the minor dominant, on G minor chord, is followed by a G major chord that functions as dominant (D). What follows next, when B_2 begins, is also a dominant chord (D/s). Thus, the transition from B_1 to B_2 is marked by a unique harmonic event: the evaded cadence of the unexpected resolution of the two dominant chords D-D/s between the fourth beat of m. 8 and the first beat of m. 9. This harmonic event marks the division of the entire piece into two sections $A+B_1$ and B_2+B_3 related by the proportion 1:1.

B_2 (mm. 9-12) preserves a constant harmonic rhythm of half-value time ($HR=1:2^1$). The harmonic relationships and the melodic elements formulate an internal structure that develops in two stages (Figure 1). In the first stage (mm. 9-10), a one-measure structural unit, defined in m. 9, constitutes the model of a diatonic imitative sequence. The first exposition of this model is realized in the area of the subdominant (s); the model is then repeated in the area of the parallel tonic (tP) in m. 10. The internal structure indicates the relation 1:1. In the second stage (mm. 10-11) a new sequence begins; the model is based on a half-measure structural unit, defined on the first and second beats in m. 11. Three repetitions of this model lead from the subdominant chord in first inversion (s_3) to the subdominant chord in root position (s) on the first beat of m. 12, through a descending step motion. The sequence of these events is interrupted on the third beat of m. 12.[xi] Both voices now define a half-measure structural unit on the dominant chord that will establish the tonic

on the first beat of m. 13. Consequently, the internal structure demonstrates the relation 3:1. B_2 begins in the tonal area of the subdominant in m. 9, moves for a while to the parallel tonic in m. 10, and returns to the tonic in m. 11.

The recapitulation consists of B_3 (mm. 13-16). The changes of the harmonic rhythm point out an internal structure which demonstrates the relation 3:1 (Figure 1). Particularly, in mm. 13-15, the harmonic rhythm remains constant in half-value time (HR=$1:2^1$). Nevertheless, the progression of the tonal functions indicate a one-measure structural unit, defined in m. 13 by the succession t-D, which is repeated twice in other forms, tG-D in m. 14 and D/s-D in m. 15. B_3 is mostly articulated by the successive appearances of various forms of the prime motive which intensify the repetitions of the structural unit. The final measure of the piece, m. 16, as the last distinct unit partially demonstrates CHR_2. Hence, the piece ends in the same manner that the exposition ended before. The relation 3:1 and the melodic elements of this section refer directly to the exposition.

B_3 demonstrates also the relation 1:1 between mm. 13-14 and mm. 15-16. This division is associated to the harmonic event of the unexpected resolution of the two dominant chords (D-D/s) that was mentioned above. This event re-appears here, between the second half of m. 14 and the first half of m. 15, as a remembrance of its presence in the development. Its unique quality establishes in the final part of the piece the relation 1:1. Consequently, B_3 brings up elements that refer to the entire piece.

Form and structure

According to the analysis presented above, the construction of the piece is based on well designed structural units. The origins of all structural units are in part A. Every unit is derived by a removal which, in general, maintains the tonal functions of chord progressions and the melodic material attached to them. The quality features of the structural units will be discussed later; at the moment let us focus on the time space that they occupy.

Structural rhythm

The alteration of the succession of the structural units follows a pattern (Figure 2a). Thus, the changes in the structural rhythm (CSR) are as shown in Table 2.

Part	Measure/ time unit	Structural unit	Duration	CSR
A	1/1	1	16	1:16 or $1:2^4$
B_1	5/17	1	8	1:8 or $1:2^3$
B_2	9/33	1	4	1:4 or $1:2^2$
	11/41	1	2	1:2 or $1:2^1$
B_3	13/49	1	4	1:4 or $1:2^2$

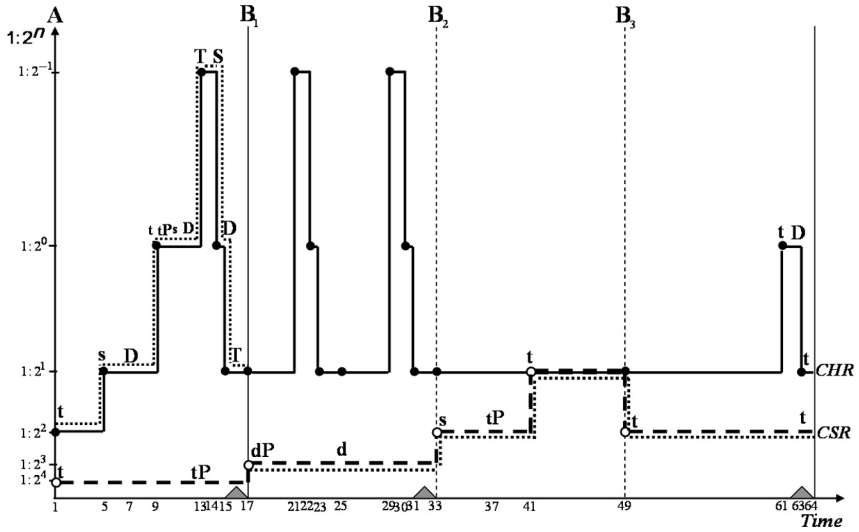
Table 2. The changes of the structural rhythm, CSR, in the overall form of the piece.

Then, CSR is represented by the following terms $1:2^4$, $1:2^3$, $1:2^2$, $1:2^1$, $1:2^4$ of the function $f(n)=1:2^n$, where n is some integer. From this series, the first term $1:2^4$ that corresponds to part A will be discussed later; let us concentrate on the terms related to part B. The set of these terms is divided into two subsets, CSR_1 = {$1:2^3$, $1:2^2$, $1:2^1$} and CSR_2 = {$1:2^1$, $1:2^2$} that have $1:2^1$ in common. In CSR_1 each term leads to the following one by multiplication by 2; thus, each term is related to the following one by the proportion 1:2. The proportional increase reaches its maximum ($1:2^1$) in m. 11 and results to a proportional acceleration of the structural rhythm. In CSR_2 each term leads to the following one by division by 2 and consequently each term is related to the following one by the proportion 2:1. This proportional decrease results to a proportional retardation of the structural rhythm. Additionally, when part B is repeated, retardation continues to the

term $1:2^3$, thus, the first term in CSR_1. Consequently, the retardation of CSR_2 leads to the acceleration of CSR_1, as if the two processes define a complete circle. This means that CSR in part B is an underlying cyclic process that comprises the proportions 1:2 and 2:1 as a result of the displacement of the structural units.

Moreover, the durations of the individual structural units of the terms of CSR_1, $1:2^3$, $1:2^2$, and of the terms of CSR_2, $1:2^1$, $1:2^2$, are proportionally related by 2:1 under retrogression (Figure 2b). Particularly, $1:2^3$ is related to $1:2^2$, and $1:2^2$, is related to $1:2^1$. The relation 2:1 under retrogression intensifies the cyclic progression of SR. It refers to two things: the relation between the acceleration and the retardation of the structural rhythm and the return back to the tonic.

(a)



(b)

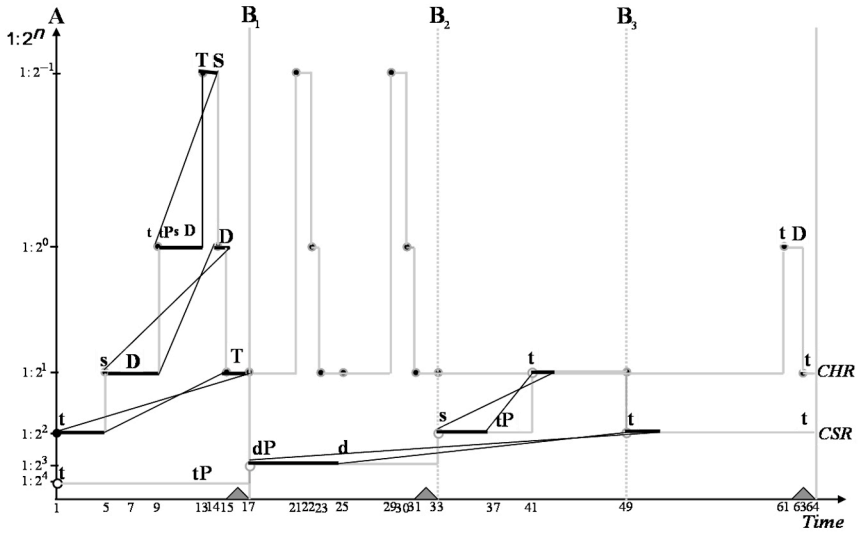


Figure 2. Graphical representation of: (a) how CHR, along with the tonal functions, and CSR, along with the tonal areas, follow time paths within the formal sections of the piece; (b) how the durations of the terms of CHR in part A are related to each other and the durations of the individual structural units of the terms of CSR are related to each other, by the proportion 2:1 under retrogression. The gray triangles in the axis of time represent the three major cadences of the piece.

Form and proportion

The three types of proportional relations 3:1, 1:2, 2:1, and 1:1, exposed in part A, are reflected to the structure of the entire piece as shown in Table 3. Specifically, 3:1 and 1:1 are reflected to the internal structure of the individual parts. 3:1, 1:1, and 1:3, are also reflected to the large-scale structure of the overall form. Finally, 1:2 and 2:1 are reflected to the relation of the formal sections of the work (Figures 1 and 2a).

Proportions of form: 1:3, 3:1, 1:1

The piece demonstrates the proportion 1:3 as the first relation between the two parts A and $B_1+B_2+B_3$; A=4 measures and $B_1+B_2+B_3=12$ measures. Moreover, the evolvement of the processes HR and SR prompts the relation 3:1 between $A+B_1+B_2$ and B_3. $A+B_1+B_2$ may be considered as a large-scale formal entity in which a significant process is completed: the proportional acceleration of the structural rhythm (CSR_1) ends at the *maximum* in m. 12. This process has also been accompanied by the removal of the tonal space from the tonic to other spaces and the return back to it. The final return is confirmed by the cadence on the first beat of m. 13 when B_3 is about to begin. This cadence establishes the tonic for the first time inside the piece and it takes place on the strongest beat of the measure while all other cadences took place on the weaker third beat. The final return to the tonic is also marked by the appearance of the prime motive at the beginning of B_3 in the lower voice.

The overall form demonstrates also the relation 1:1 between $A+B_1$ and B_2+B_3. Sections A and B_1 have many things in common. Their structural proximity lays primarily on the fact that B_1 consists of a significant part of A. Moreover, the tonal areas of these sections are closely associated. Part A moves in the tonal space of the tonic (t) and B_1 in the minor dominant (d). These two spaces are a perfect fifth up apart. Additionally, in the internal structure of A, the pitch-material is transferred from the area of the tonic (t) to the area of the relative key (tP), which is a minor third up; similarly, in the internal structure of B_1 the tonal areas are the parallel dominant (dP) and the minor dominant (d), which is a minor third down. Thus, the tonal areas of the two sections are associated by transposition and retrograde inversion.

<table>
<tr><th></th><th>Exposition</th><th colspan="3">Development</th><th>Recapitulation</th></tr>
<tr><td>Parts</td><td>A</td><td>B_1</td><td colspan="2">B_2</td><td>B_3</td></tr>
<tr><td>Measures</td><td>1-4</td><td>5-8</td><td>9-10</td><td>11-12</td><td>13-16</td></tr>
<tr><td colspan="6">Individual sections</td></tr>
<tr><td>Internal structure</td><td>3:1</td><td></td><td></td><td>3:1</td><td>3:1</td></tr>
<tr><td></td><td>2:1</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>1:1</td><td>1:1</td><td>1:1</td><td></td><td>1:1</td></tr>
<tr><td colspan="6">Proportions of form</td></tr>
<tr><td>Overall form</td><td colspan="5">$A:B_1+B_2+B_3=1:3$</td></tr>
<tr><td></td><td colspan="5">$A+B_1+B_2:B_3=3:1$</td></tr>
<tr><td></td><td colspan="5">$A+B_1:B_2+B_3=1:1$</td></tr>
<tr><td>Formal sections</td><td colspan="5"></td></tr>
<tr><td>Exposition and development</td><td colspan="3">$A:B_1+B_2=1:2$</td><td colspan="2" rowspan="3"></td></tr>
<tr><td></td><td colspan="3">$A+B_1:B_2=2:1$</td></tr>
<tr><td></td><td></td><td colspan="2">$B_1+B_2 = 3:1$</td></tr>
<tr><td>Development and recapitulation</td><td rowspan="2"></td><td colspan="4">$B_1:B_2+B_3=1:2$</td></tr>
<tr><td></td><td colspan="4">$B_1+B_2:B_3=2:1$</td></tr>
</table>

Table 3. The structural proportions of the individual sections, the overall form and the formal sections of the piece.

Similarly, sections B_2 and B_3 share common features. The harmonic rhythm remains constant, at half-value time, until about the end of the piece. In $A+B_1$ the overall transposition of the tonal space is a perfect fifth up; the re-transition to the area of the tonic, accomplished in the next entity, B_2+B_3, is realized by the transposition from the space of the subdominant (s) to the space of the tonic (t), thus, a perfect fifth up. Consequently, the two entities appear a direct analogy. Moreover, the unique harmonic event that appears at the point where the two entities meet each other, the fourth beat of m. 8 to the first beat of m. 9, the succession of two dominant chords (D-D/s), subsequently divides the piece into two equal sections. Remarkably, this unexpected resolution comes at a certain moment in the piece. It is noticeable that number 8 is the arithmetic mean of 4 and 12 which are the numbers of measures of part A and part B correspondingly. Furthermore, the re-appearance of the prime motive in the beginning of m. 9 recalls here the opening of the piece; it seems that, in its beginning, B_2 resembles A. The picture prompt by these observations is that of a two-section piece in which the two sections are related to each other with the proportion 1:1.

Proportions of formal sections: 1:2 and 2:1

Exposition and development constitute a separate entity that demonstrates the proportional augmentation of CSR_1. Additionally, significant features of its internal structure result in an internal division according to which two proportional interrelations coexist: the relation 1:2 between A and B_1+B_2 and the relation 2:1 between $A+B_1$and B_2.

According to the measure numbers A and B_1+B_2 are related by 1:2, since A=4 measures and B_1+B_2=8 measures. More important, though, is the relation 2:1. In respect to CHR_1, the internal structure of $A+B_1$ is 3:1+1:1 laid out in 8 measures. Similarly, the harmonic and structural units in B_2 create the internal structure 1:1-3:1 laid out in 4 measures. These two structures are related to each other by the proportion 2:1 under retrogression. As mentioned before, in $A+B_1$ the overall transposition of the tonal space is a perfect fifth up, from the space of the tonic (t) to the space of the minor dominant (d). B_2 demonstrates the same transposition of a perfect fifth up, from the space of the subdominant (s) to that of the tonic (t).

Development and recapitulation constitute part B. Features similar to those discussed above, indicate that two proportional interrelations coexist: the relations B_1:B_2+B_3=1:2 and B_1+B_2:B_3=2:1.

According to the measure numbers B_1 and B_2+B_3 are related by 1:2, since B_1=4 measures and B_2+B_3=8 measures. Also, in respect to CHR, B_2 and B_3 are similarly structured, since they preserve a constant harmonic rhythm; the internal structure of B_1 is 1:1 and of B_2+B_3 is 1:1-3:1+1:1/3:1, thus B_3 incorporates in its structure the structural relations of B_2.

Moreover, B_1+B_2 constitute that part of the piece in which the proportional acceleration of the structural rhythm (CSR_1) begins and ends. The relation B_1+B_2:B_3 as 2:1 is accompanied by the final return to the tonic, confirmed by the cadence on the first beat of m. 13, and the appearance of the prime motive at the beginning of B_3 in the lower voice.

This discussion shows that the three types of proportional relations 3:1, 1:2, 2:1, and 1:1 exposed in part A actually operate on the overall form and the formal sections in both time directions, forward and backward.[xii] Inevitably, a question rises: what kind of formal process is that which allows for all these relationships to exist in both time directions? How is it accommodated in the binary form of the piece?

Formal process

The two processes presented above, CHR and CSR, appear to have direct analogies: they clearly relate to the formal sections of the piece, A and B; they move from

acceleration to retardation under 1:2 and 2:1 correspondingly; they are both cyclic processes and this is enacted when A and B are repeated; they embody the relation 2:1 under retrogression, and this is somehow related to the move away from the tonic and the return back to it. Consequently, a logical conclusion is that CHR_1 maps on to CSR_1 under the relation 4:1 as shown in Table 4; indeed, the relation 4:1 appears to interpret the close connection of the two processes.

There is, however, a certain point in the piece that constitutes a kind of structural inconsistency. If SR operated in the overall form as HR operates in part A, then CSR should have reached the maximum in the first beat of m. 13, just as CHR reaches the critical point in the first beat of m. 4, that corresponds to time unit 13. This does not happen. Instead, CSR reaches the maximum sooner, in the first beat of m. 11, and lasts 8 time units, until the first beat of m. 13. Also, this one to one map does not include section B_3; so, how does this section appear? Moreover, if CHR_1 mapped on to CSR_1 in such an obvious manner, what would happen with the relation between CHR_2 and CSR_2? Concerning the organization of the overall form these questions seemed to be unanswered. Nevertheless, there is an element that connects the two processes and unifies the results of the present analysis: the durations of the harmony-events of HR and of the structure-events of SR.

Part	Time unit	CHR_1	Part	Measure	CSR_1
A	1	$1:2^2$	A	1	$1:2^4$
	5	$1:2^1$	B_1	5	$1:2^3$
	9	$1:2^0$	B_2	9	$1:2^2$
	13	$1:2^{-1}$	B_2	11	$1:2^1$

Table 4. One to one map between CHR_1 and CSR_1 under the scaling transformation 4:1.

Harmonic rhythm and structural rhythm: 1:8

HR and SR are cyclic processes that operate in the formal sections of the piece A and B in an analogous manner featured by qualitative characteristics. These two orbits of events touch each other under a scale transformation 1:8 realized by retrogression. Part B is reading part A backwards (Figure 3).

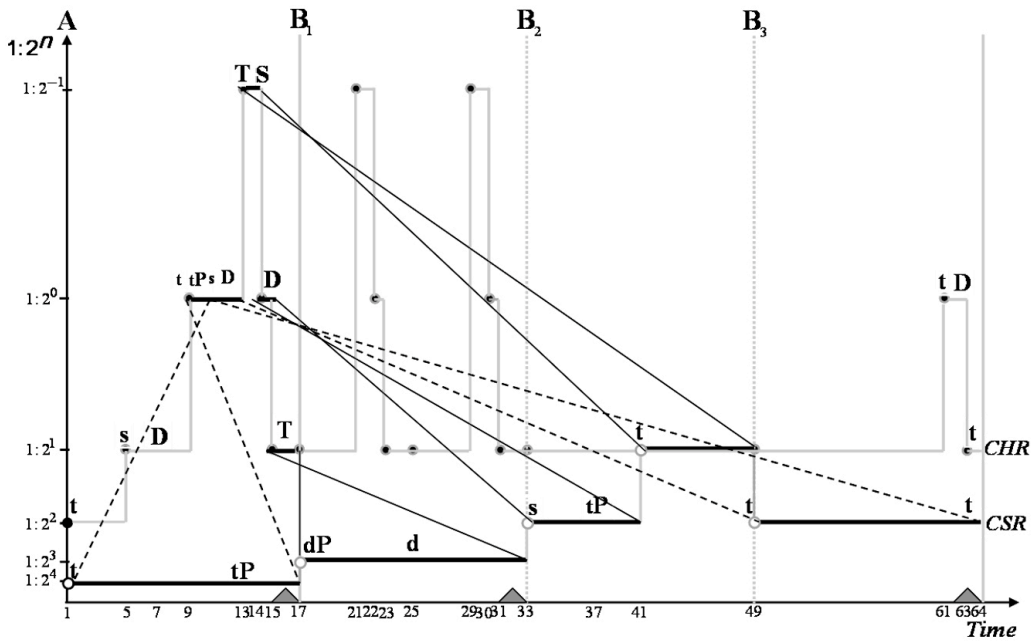


Figure 3. Graphical representation of the relation between the time paths of CHR and the time paths of CSR under the scaling transformation 1:8 under retrogression.

Part	Time unit	CHR	Duration	Part	Measures	CSR	Duration
A	15-16	$1:2^1$	2	B_1	5-8	$1:2^3$	16 (2)
	14	$1:2^0$	1	B_2	9-10	$1:2^2$	8 (1)
	13	$1:2^{-1}$	1		11-12	$1:2^1$	8 (1)
	12-11	$1:2^0$ (1/2)	2	B_3	13-16	$1:2^2$	16 (2)
	10-9	$1:2^0$ (1/2)	2	A	1-4	$1:2^4$	16 (2)

Table 5. One to one map between CHR and CSR under the scaling transformation 1:8 under retrogression.

The durations, in time units, of the harmony-events of CHR_2 in part A (time units 13-16) are 1-1-2 and they are proportionally related by 1:1:2; the durations, in time units, of the structure-events of CSR_1 in part B are 16-8-8, thus they are also proportionally related by 2:1:1. This series of proportions exists nowhere else in the piece. Consequently, it indicates that the durations of the structure-events of CSR_1 in part B reflect the durations of the harmony-events of CHR_2 in part A, under retrogression. Particularly, the duration of the third term of CHR_2, $1:2^1$, in time units 15-16, is related to the duration of the first term of CSR_1, $1:2^3$, in mm. 5-8; similarly, $1:2^0$, in time unit 14, is related to $1:2^2$, in mm. 9-10; $1:2^{-1}$, in time unit 13, is related to $1:2^1$, in mm. 11-12, as shown in Table 5. Consequently, the time paths of the above terms map on to one another under the relation 1:8.

Furthermore, this backward-reading process might well be continued to the following term of CHR, the term $1:2^0$. The duration of the harmony-events related to this term is 4 time units, but it may well be considered as 2+2, representing the time units 12-11 and 10-9 correspondingly. Then, the duration of the one half of the term $1:2^0$ of CHR_1, in time units 12-11, maps on the duration of the term $1:2^2$ of CSR_2, in mm. 13-16, under the scaling transformation 1:8; similarly, the duration of the other half of the term $1:2^0$ of CHR_1, in time units 10-9, maps on the duration of the term $1:2^4$ of CSR_1, in mm. 1-4, under the scaling transformation 1:8. Thus, the backward reading of part A, on the one hand, creates the time spaces in part B and, on the other hand, maps part A into itself.[xiii]

This interpretation suggests that the exposition of the process HR in part A is followed by a long process of retrogression under the relation 1:8 which results to the generation of SR in part B and ultimately returns to part A where it begun. There are two interesting observations that intensify this perspective (Figure 1). First, it absolutely explains why CSR reaches the maximum in m. 11 and why it lasts 8 time units: according to the backward reading, m. 11 in the overall form of the piece corresponds exactly to the critical point of CHR in time unit 13; since the critical point of CHR lasts 1 time unit and the scaling transformation is realized under the relation 1:8, the maximum of CSR should last 8 time units and this happens. Second, it also explains why the proportion 3:1, connected to the critical point of CHR in part A, is expressed as 1:3 in the overall form of the piece, between parts A and B; thus, it provides a strong criterion for the interpretation of the overall form.

Harmonic reflections

At the same time, one more backward-reading process takes place; the reflection of a series of tonal functions in HR_1 to a series of tonal spaces-areas in SR under retrogression (Figure 1, black triangle markings). Starting from the critical point of CHR_1, the term $1:2^{-1}$ and moving backwards, this reflection is realized as follows. The major tonic function in time unit 13 (the E flat major chord) is literally the parallel tonic, tP, which refers also to the minor tonic, t, in the beginning of the work; this function is reflected to the overall tonal space t in mm. 1-4. Similarly, the dominant function in time unit 12 (the B flat major chord) is literally the parallel dominant area, dP, in time units 17-24, which refers also to the area of the coming minor dominant, d, in time units 25-32;this

Part	Time unit	CHR	Tonal function	Part	Measures	Time unit	CSR	Tonal space	Tonal area
A				A	1-3	1	$1:2^4$	t	t
	13	$1:2^{-1}$	T		4	13			tP
	12	$1:2^0$	D	B_1	5-6	17	$1:2^3$	d	dP
					7-8	25			d
	11	$1:2^0$	s	B_2	9	33	$1:2^2$		s
	10	$1:2^0$	tP		10	37	$1:2^1$		tP
	9	$1:2^0$	t		11	41	$1:2^2$		t

Table 6. The reflections of the tonal functions during the moving away of the tonic to the tonal spaces/areas during the returning back to it, under retrogression.

function is reflected to the overall tonal space d in mm. 5-8. The subdominant function, s, in time unit 11 is reflected to the subdominant tonal area, s, in time units 33-36; the parallel tonic function, tP, in time unit 10 is reflected to the parallel tonic area, tP, in time units 37-40; finally, the tonic function, t, in time unit 9 is reflected to the tonic area, t, in time unit 41. These reflections are shown in Table 6.

It is worth it to point out that the series of tonal functions from time unit 9 to time unit 13 represents the intermediate space of the transition from the area of the tonic to the area of the parallel tonic in part A, at the critical point of CHR; thus, they indicated the way of moving away from the tonic in the beginning of the piece. Through the reflections described above, the same series, in retrograde order, outlines also the way of returning back to the tonic in part B, at the moment when SR reaches the maximum.

It is also remarkable that the points of the departure from the tonic, the time unit 9 in A, and of the arrival back to it, the time unit 41 in B, are not accidental (Figure 1, up-arrow markings); both fall on to numbers that represent the arithmetic means of the time spaces of the correlated sections. More specifically, considering that part A consists of two areas related by 3:1, the first occupies the space of time units 1-12, thus 12, and the second occupies the space of time units 13-16, thus 4. The arithmetic mean of numbers 12 and 4 is number 8; in part A, number 8 falls on to time unit 9. Similarly, considering that part B consists of the development B_1+B_2 and the recapitulation B_3, the time space occupied by each section is 32 and 16 time units correspondingly. The arithmetic mean of numbers 32 and 16 is number 24; in part B, number 24 falls on to time unit 41. Moreover, the time unit 41 divides the development, B_1+B_2, into two sections of measure numbers related by the proportion 3:1. Thus, the maximum of SR recreates in the development the critical point of CHR_1 that prompted the relation 3:1 in the exposition.

Up to this point, the analysis has demonstrated how part B originates from part A by utilizing a significant amount of the structural and harmonic materials that were exposed in the opening of the work, through retrogressive processes of creating time space and of reflecting tonal functions to tonal spaces and areas.

Structural reflections

A third process, realized forward in time, that exhausts the rest of the materials of part A is the construction of the structural units of part B by the displacement of certain harmonic units and melodic elements (Figure 1, lines, slurs, and rectangles on the music). Particularly, the harmonic progression s-D and the melodic material in time units 5-8, along with the harmonic progression T-S-D-T in time units 13-16, constitute the structural unit in B_1 (time units 17-24), repeated in the areas dP and d. Similarly, the progression D-t-tP and the melodic material in time units 7-10 constitute the structural unit in B_2 (time unit 33-36), repeated in the areas s and tP.[xiv] The tonal functions s in time unit 11

and 13, along with the melodic material attached to them, constitute the structural unit of the imitative sequence in B_2 (time unit 41-46); in the same manner, the tonal functions D in time units 12 and 14, along with the melodic material attached to them, constitute the final structural unit in B_2 (time units 47-48). Finally, the function of major tonic in time unit 13 is reflected to the tonic confirmed in B_3 (time unit 49) where the musical material returns to the beginning. Consequently, as the structural units become smaller, they end to approach the tonal function of their origins.

The interpretation of the construction of the structural units through displacement brings up interesting observations. First, the functions s, D, in time units 11+13 and 12+14 respectively, that are connected to the final approach and the moving away of the critical point of HR in part A are reproduced in B_2 in time units 41-48 on the maximum of SR; thus, the acceleration and retardation of HR in the exposition rhymes with the acceleration of SR in the development. Second, the function s in time unit 11 and the function s in time unit 41 fall on to numbers that represent the harmonic means of the time spaces of the correlated sections. As mentioned above, part A consists of two areas related by 3:1, with time units 12 and 4. The harmonic mean of numbers 12 and 4 is number 6; in part A, number 6 falls on to time unit 11 by counting backwards. Similarly, the entire form consists of the two parts A and B with measure numbers 4 and 12. The harmonic mean of numbers 4 and 12 is number 6; in the overall form AB, number 6 falls on to m. 11 by counting backwards and this point corresponds to time unit 41. Third, the function of major tonic in time unit 13 is associated to the critical point of the acceleration of HR that prompts the relation 3:1 in part A. Its reflection to the tonic confirmed in B_3 in time unit 49 recreates in the overall form the same proportion: 3:1 between $A+B_1+B_2$ and B_3.

Harmonic rhythm and cadences: 1:1

Perfect cadences appear at certain moments in the evolution of the work.[xv] There are three major cadences realized on the third beat of the measure where they are located. The perfect cadence in E flat major (tP), comes in m. 4; the perfect cadence in G minor (d) is located in m. 8; the last cadence in C minor (t) comes to the end. In relation to the overall time of the work, measured in sixteen measures, the cadences are placed as follows: 4:16 or $1:2^2$, 8:16 or $1:2^1$, 16:16 or $1:2^0$. Hence, the occurrences of the cadences can be represented by the following terms $1:2^2$, $1:2^1$, and $1:2^0$. Interestingly, these occurrences reflect on the entire time of the piece the first three terms of CHR_1.

To conclude, the formal process comprises two cyclic processes of events: the harmonic rhythm in part A and the structural rhythm in part B, moving in time under the relations 1:2 (acceleration), 2:1 (retardation), and 2:1 under retrogression. The structural rhythm process is the outcome of the retrogression of the harmonic rhythm process under a scaling transformation 1:8. The retrogression completes three actions: it generates part B, it turns the structural rhythm into the heart of part A, and it reflects the individual harmonic functions to tonal spaces and areas. During the change of the structural rhythm, the successive structural units in part B are generated by the displacement of the harmonic units of part A. This is associated to the proportional relations manifested in several structural levels of the piece and with the fact that all relations operate in both time directions, forward and backward. This interpretation explains the accommodation of these processes in the binary form, justifies the appearance of moments-points of symmetry, and demonstrates the role of part A as exposition and of part B as development and recapitulation as followed in the description throughout the above discussion.

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[i] Tinctoris (c. 1475, reprint 1979: p. 3).

[ii] Schillinger (1966: p. 399).

[iii] The transition from the contrapuntal texture to the harmonic background is aligned with what Wolff calls Bach's musical Credo: Harmony is constituted by accumulated counterpoint. It defines both triadic chord and contrapuntal function (Wolff, 2007: p. 100).

[iv] These points are marked by a variety of musical elements such as harmonic, structural and melodic, thus, they appear in complete accordance to the actual music.

[v] Studying the points of changes is a way to escape the predetermined value of the measure and to observe the underlying processes of the work. The space of the measure, however, is a definite space in Bach's time (Tatlow, 2007). So, proportions in underlying processes are inescapably manifested in the numbers of measures.

[vi] Lester refers to this piece as a two-reprise movement with four parallel sections (Lester, 2001: p. 87).

[vii] Bach, J. S. Franzosische Suiten. Frankfurt: C. F. Peters, 1951.

[viii] Due to the limited space of this paper, the intermediate steps of the reduction are not included here.

[ix] The move away from the tonic is one of the signal features motivating large-scale binary forms in the later eighteenth and nineteenth centuries (Lester, 2001: p. 83).

[x] The references to part A as exposition and to part B as development and recapitulation might seem inappropriate at this point of the analysis; as it will be demonstrated below, however, they apply perfectly to the role of these parts in the piece.

[xi] The interruption takes place in the lower voice while the upper one is adjusted by continuing the repetition of the melodic model transposed a second lower of the expected pitch.

[xii] This implies that if we forget for a while the double bar at the end of part A, the four major sections of the piece are laid out in time as a continuum of great homogeneity. Could we then find out where to put the double bar?

[xiii] As part A maps into itself, the space of the piece that remains untouched by the backward reading process is the space of mm. 1-2, as if it represents an initial, basic space from where all other things originate. This may also imply that the piece actually originated from this minimum time space, something that alludes to Schoenberg's basic idea (See also Lester, 2001: p. 52).

[xiv] At this point, time unit 33, Bach enforces the impression of the opening by bringing the interval of P8 in the lower voice.

[xv] Cadences play a significant role as sectional articulators and in Bach's music demarcate the overall tonal progression of a composition (Lester, 2001: pp. 77-78).